

Multiplicative Zagreb indices of k -trees

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The first generalized multiplicative Zagreb index of G is $\prod_{1,c}(G) = \prod_{v \in V(G)} d(v)^c$, for a real number $c > 0$, and the second multiplicative Zagreb index is $\prod_2(G) = \prod_{uv \in E(G)} d(u)d(v)$, where $d(u), d(v)$ are the degrees of the vertices of u, v . The multiplicative Zagreb indices have been the focus of considerable research in computational chemistry dating back to Narumi and Katayama in 1980s. In this paper, we generalize Narumi-Katayama index and the first multiplicative index, where $c = 1, 2$, respectively, and extend the results of Gutman to the generalized tree, the k -tree, where the results of Gutman are for $k = 1$. Additionally, we characterize the extremal graphs and determine the exact bounds of these indices of k -trees, which attain the lower and upper bounds.

Keywords: Multiplicative Zagreb indices, k -trees

1 Introduction

Throughout this paper $G = (V, E)$ is a connected finite simple undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $|G|$ or $|V|$ denote the cardinality of V . For $S \subseteq V(G)$ and $F \subseteq E(G)$, we use $G[S]$ for the subgraph of G induced by S , $G - S$ for the subgraph induced by $V(G) - S$ and $G - F$ for the subgraph of G obtained by deleting F . Let $w(G - S)$ be the number of components of $G - S$, and S be a cut set if $w(G - S) \geq 2$. For a vertex

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$v \in V(G)$, the neighborhood of v is the set $N(v) = N_G(v) = \{w \in V(G), vw \in E(G)\}$, and $d_G(v)$ (or briefly $d(v)$) denotes the degree of v with $d_G(v) = |N(v)|$. We use $G \cong H$ to denote that G is isomorphic to H and $G \not\cong H$ to denote that G is not isomorphic to H . Let K_n, P_n and S_n denote the clique, the path and the star on n vertices, respectively. In particular, we say K_n is a k -clique for $n = k$.

In the 1980s, Narumi and Katayama [7] considered the product

$$NK = \prod_{v \in V(G)} d(v)$$

which is the "Narumi-Katayama index". And recently, Todeschini and Gutman et al [4, 10, 11] studied the first and second multiplicative Zagreb indices defined as follow:

$$\begin{aligned}\Pi_1(G) &= \prod_{v \in V(G)} d(v)^2, \\ \Pi_2(G) &= \prod_{uv \in E(G)} d(u)d(v).\end{aligned}$$

Obviously, the first multiplicative Zagreb index is just the square of the NK index. Gutman [4] in 2011 characterized the multiplicative Zagreb indices for trees and determined the unique trees that obtained maximum and minimum values for $\Pi_1(G)$ and $\Pi_2(G)$, respectively.

Theorem 1 (Gutman 2011) *Let $n \geq 5$ and T_n be any tree with n vertices, then*

$$\begin{aligned}(i) \quad &\Pi_1(S_n) \leq \Pi_1(T_n) \leq \Pi_1(P_n); \\ (ii) \quad &\Pi_2(P_n) \leq \Pi_2(T_n) \leq \Pi_2(S_n).\end{aligned}$$

In this paper, we consider the first generalized multiplicative Zagreb index defined in (1) below and the second multiplicative Zagreb index: for any real number $c > 0$,

$$\begin{aligned}(1) \quad &\Pi_{1,c}(G) = \prod_{v \in V(G)} d(v)^c; \\ (2) \quad &\Pi_2(G) = \prod_{uv \in E(G)} d(u)d(v).\end{aligned}$$

Eventually, for $c = 1, 2$, (1) is just the NK index and the first multiplicative Zagreb, respectively. For (2), it is easy to see that $\Pi_2(G) = \prod_{v \in V(G)} d(v)^{d(v)}$. Also we will find the bounds of the values of $\Pi_{1,c}(G)$, $\Pi_2(G)$ for k -trees, respectively, and determine the extremal graphs which attain the bounds. Our main results are as follows:

Theorem 2 *Let T_n^k be a k -tree on $n \geq k$ vertices, then*

$$\Pi_{1,c}(S_{k,n-k}) \leq \Pi_{1,c}(T_n^k) \leq \Pi_{1,c}(P_n^k),$$

the left-side and the right-side equalities are reached if and only if $T_n^k \cong S_{k,n-k}$ and $T_n^k \cong P_n^k$, respectively.

Theorem 3 Let T_n^k be a k -tree on $n \geq k$ vertices, then

$$\Pi_2(P_n^k) \leq \Pi_2(T_n^k) \leq \Pi_2(S_{k,n-k}),$$

the left-side and the right-side equalities are reached if and only if $T_n^k \cong P_n^k$ and $T_n^k \cong S_{k,n-k}$, respectively.

2 Preliminary

It is commonly known that the class of k -trees is an important subclass of trangular graphs. Harry and Plamer [5] first introduced the 2-tree in 1968, which is showed to be maximal outerplanar graphs in [3, 6]. Beineke and Pippert [1] gave the definition of k -trees in 1969. Relating to k -trees, there are many interesting applications to the study of a computational complexity and the intersection between graph theory and chemistry [2, 9]. We will just give some notations and definitions below.

Notation 1. Let $[a, b]$ be the set of all the integers between a and b with $a \leq b$ including a, b , where a, b are integers. Also, let $(a, b] = [a, b] - \{a\}$ and $[a, b) = [a, b] - \{b\}$. In particular, $[a, b] = \emptyset$ for $a > b$.

Notation 2. For any integer p , if $p \geq 0$, we denote $x_{\max\{0,p\}} = x_p$; If $p < 0$, we say $x_{\max\{0,p\}}$ does not exist.

Definition 1. The k -tree, denoted by T_n^k , for positive integers n, k with $n \geq k$, is defined recursively as follows: The smallest k -tree is the k -clique K_k . If G is a k -tree with $n \geq k$ vertices and a new vertex v of degree k is added and joined to the vertices of a k -clique in G , then the obtained graph is a k -tree with $n + 1$ vertices.

Definition 2. The k -path, denoted by P_n^k , for positive integers n, k with $n \geq k$, is defined as follows: Starting with a k -clique $G[\{v_1, v_2 \dots v_k\}]$. For $i \in [k + 1, n]$, the vertex v_i is adjacent to vertices $\{v_{i-1}, v_{i-2} \dots v_{i-k}\}$ only.

Definition 3. The k -star, denoted by $S_{k,n-k}$, for positive integers n, k with $n \geq k$, is defined as follows: Starting with a k -clique $G[\{v_1, v_2 \dots v_k\}]$ and an independent set S with $|S| = n - k$. For $i \in [k + 1, n]$, the vertex v_i is adjacent to vertices $\{v_1, v_2 \dots v_k\}$ only.

Definition 4. A vertex $v \in V(T_n^k)$ is called a k -simplicial vertex if v is a vertex of degree k whose neighbors form a k -clique of T_n^k . Let $S_1(T_n^k)$ be the set of all k -simplicial vertices of T_n^k , for $n \geq k + 2$, and set $S_1(K_k) = \emptyset, S_1(K_{k+1}) = \{v\}$, where v is any vertex of K_{k+1} . Let

$G = G_0, G_i = G_{i-1} - v_i$, where v_i is a k -simplicial vertex of G_{i-1} , then $\{v_1, v_2 \dots v_n\}$ is called a simplicial elimination ordering of the n -vertex graph G .

Definition 5. If $w(G - S) \leq 2$ for any k -clique $G[S]$ of T_n^k , we say T_n^k is a hyper pendent edge; If there exists a k -clique $G[S]$ with $w(G - S) \geq 3$, let C be a component of $T_n^k - S$ and contain a unique vertex belonging to $S_1(G)$, then we say that $G[V(S) \cup V(C)]$ is a hyper pendent edge of T_n^k , denoted by \mathcal{P} . In particular, a k -path is a hyper pendent edge.

Moreover, let $G[\{v_1, v_2 \dots v_k\}]$ denote the initial k -clique, then just by the definition of k -trees, one can get

Fact 1. For the k -star, the degree of vertex v_i can be characterized as follows: $d(v_i) = n - k$, for $i \in [1, k]$; $d(v_i) = k$, for $i \in [k + 1, n]$.

Fact 2. For the k -path, the degree of vertex v_i can be characterized as follows:

- (1) If $4 \leq n \leq 2k$, $d(v_i) = k + i - 1$, for $i \in [1, n - k - 1]$; $d(v_i) = n - 1$, for $i \in [n - k, k + 1]$; $d(v_i) = k + n - i$, for $i \in [k + 2, n]$.
- (2) If $n \geq 2k + 1$, $d(v_i) = k + i - 1$, for $i \in [1, k]$; $d(v_i) = 2k$, for $i \in [k + 1, n - k]$; $d(v_i) = k + n - i$, for $i \in [n - k + 1, n]$.

Easily verified through induction by using the above observations, one can deduce the first generalized multiplicative Zagreb indices and second multiplicative Zagreb indices of the k -path and k -star as follows.

Fact 3. Let $S_{k, n-k}$ be a k -star on $n \geq k + 1$ vertices, then

- (1) $\Pi_{1,c}(S_{k, n-k}) = (n - k)^{ck} k^{c(n-k)}$;
- (2) $\Pi_2(S_{k, n-k}) = (n - k)^{k(n-k)} k^{k(n-k)}$.

Fact 4. Let P_n^k be a k -path on $n \geq k + 1$ vertices, then

- (1.1) $\Pi_{1,c}(P_n^k) = (n - 1)^c \prod_{i=k}^{n-2} i^{2c}$, if $n \in [k + 1, 2k]$;
- (1.2) $\Pi_{1,c}(P_n^k) = (2k)^{c(n-2k)} \prod_{i=k}^{2k-1} i^{2c}$, if $n \geq 2k + 1$;
- (2.1) $\Pi_2(P_n^k) = (n - 1)^{n-1} \prod_{i=k}^{n-2} i^{2i}$, if $n \in [k + 1, 2k]$;
- (2.2) $\Pi_2(P_n^k) = (2k)^{2k(n-2k)} \prod_{i=k}^{2k-1} i^{2i}$, if $n \geq 2k + 1$.

By considering the derivatives of the following functions, one can get

Fact 5. The function $f(x) = \frac{x}{x + m}$ is strictly increasing for $x \in [0, \infty)$, where m is a positive integer.

Fact 6. The function $f(x) = \frac{x^x}{(x + m)^{x+m}}$ is strictly decreasing for $x \in [0, \infty)$, where m is a positive integer.

3 Main proofs

Firstly, we give some lemmas that are critical in the proof of our main results.

Lemma 1 *For any k -tree $G \not\cong S_{k,n-k}$, let $u \in S_2$, $N(u) \cap S_1 = \{v_1, v_2 \dots v_s\}$, where $s \geq 1$ is an integer, then*

(1) *For any i with $1 \leq i \leq s$, there exists a vertex $v \in N(u) - \{v_1, v_2 \dots v_s\}$ of degree at least k in $G[V(G) - \{v_1, v_2 \dots v_s\}]$ such that $vv_i \notin E(G)$.*

(2) *There exists a k -tree G^* such that $\Pi_{1,c}(G^*) < \Pi_{1,c}(G)$ and $\Pi_2(G^*) > \Pi_2(G)$.*

Proof. For (1), let $G' = G[V(G) - \{v_1, v_2 \dots v_s\}]$ and $S = N(u) - \{v_1, v_2 \dots v_s\}$, we obtain that $d_{G'}(u) = |S| = k$ and $G[S]$ is a k -clique by $u \in S_2$. Since $G \not\cong S_n^k$, $d_{G'}(v) \geq k$ for all $v \in S$. And by the facts that $N(v_i) \subseteq (N(u) - \{v_1, v_2 \dots v_s\}) \cup \{u\}$ with $|N(v_i)| = k$ and $|(N(u) - \{v_1, v_2 \dots v_s\}) \cup \{u\}| = k + 1$, we have for any $i \in [1, s]$, there exists a vertex $v \in S$ such that $vv_i \notin E(G)$.

For (2), choose v_1 and by (1) there exists a vertex $v \in N(u) - \{v_1, v_2 \dots v_s\}$ with $d_{G'}(v) \geq k$ such that $vv_1 \notin E(G)$. If $d_{G'}(v) = k$, and by $uv \in E(G')$, we obtain G' is a $k + 1$ -clique. Let $x \in S$ be the vertex such that $d(x) = \min_{v \in S} \{d(v)\}$, and let v_t be the vertex such that $v_t x \in E(G)$, $v_t y \notin E(G)$ for some $t \in [1, s]$ and $y \in S$, that is, $d(x) - 1 < d(y)$. Construct a new graph G^* such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{v_t x\} + \{v_t y\}$. Denote $G_0 = G[V(G) - \{x, y\}]$, since $d(x) - 1 < d(y)$, and by the definition of $\Pi_{1,c}(G)$, $\Pi_2(G)$ and Fact 5, Fact 6, we have

$$\begin{aligned} \frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{[\prod_{w \in V(G_0)} d(w)^c] d(y)^c d(x)^c}{[\prod_{w \in V(G_0)} d(w)^c] [d(y) + 1]^c [d(x) - 1]^c} \\ &= \frac{d(y)^c d(x)^c}{[d(y) + 1]^c [d(x) - 1]^c} \\ &= \frac{[d(y) + 1]^c}{[d(x) - 1]^c} \\ &> 1. \end{aligned}$$

Also,

$$\begin{aligned}
\frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{[\prod_{w \in V(G_0)} d(w)^{d(w)}] d(y)^{d(y)} d(x)^{d(x)}}{[\prod_{w \in V(G_0)} d(w)^{d(w)}] [d(y) + 1]^{d(y)+1} [d(x) - 1]^{d(x)-1}} \\
&= \frac{d(y)^{d(y)} d(x)^{d(x)}}{[d(y) + 1]^{d(y)+1} [d(x) - 1]^{d(x)-1}} \\
&= \frac{\left[\frac{d(y)^{d(y)}}{[d(y) + 1]^{d(y)+1}} \right]}{\left[\frac{[d(x) - 1]^{d(x)-1}}{d(x)^{d(x)}} \right]} \\
&< 1.
\end{aligned}$$

Thus, we find that the k -tree G^* satisfies $\Pi_{1,c}(G^*) < \Pi_{1,c}(G)$ and $\Pi_2(G^*) > \Pi_2(G)$, we are done.

If $d_{G'}(v) \geq k+1$, reorder the subindices of $\{v_1, v_2 \dots v_s\}$ such that $vv_i \notin E(G)$ with $i \in [1, s_1]$, where $s_1 \leq s$, and by the fact that $G[N(u) - \{v_1, v_2 \dots v_s\}]$ is a k -clique, we have $d(u) = k+s$ and $d(v) \geq k+1+s-s_1$, that is, $d(v) \geq d(u) - s_1 + 1$. Construct a new graph G^* such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{uv_i\} + \{vv_i\}$, for all $i \in [1, s_1]$. Since $G[N(u) - \{v_1, v_2 \dots v_s\} + \{u\}]$ is a $k+1$ -clique, and for any i , $N(v_i) \subseteq N_{G-\{v_1, v_2 \dots v_s\}}(u) \cup \{u\}$, then G^* is a k -tree. Denote $G_0 = G[V(G) - \{u, v\}]$, since $d(v) \geq d(u) - s_1 + 1$, and by the definition of $\Pi_{1,c}(G)$, $\Pi_2(G)$ and Fact 5, Fact 6, we have

$$\begin{aligned}
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{[\prod_{w \in V(G_0)} d(w)^c] d(v)^c d(u)^c}{[\prod_{w \in V(G_0)} d(w)^c] [d(v) + s_1]^c [d(u) - s_1]^c} \\
&= \frac{d(v)^c d(u)^c}{[d(v) + s_1]^c [d(u) - s_1]^c} \\
&= \frac{\left[\frac{d(v)^c}{[d(v) + s_1]^c} \right]}{\left[\frac{[d(u) - s_1]^c}{d(u)^c} \right]} \\
&> 1.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{[\prod_{w \in V(G_0)} d(w)^{d(w)}] d(v)^{d(v)} d(u)^{d(u)}}{[\prod_{w \in V(G_0)} d(w)^{d(w)}] [d(v) + s_1]^{d(v)+s_1} [d(u) - s_1]^{d(u)-s_1}} \\
&= \frac{d(v)^{d(v)} d(u)^{d(u)}}{[d(v) + s_1]^{d(v)+s_1} [d(u) - s_1]^{d(u)-s_1}} \\
&= \frac{\left[\frac{d(v)^{d(v)}}{[d(v) + s_1]^{d(v)+s_1}} \right]}{\left[\frac{[d(u) - s_1]^{d(u)-s_1}}{d(u)^{d(u)}} \right]} \\
&< 1.
\end{aligned}$$

Hence, we find that the k -tree G^* satisfies $\Pi_{1,c}(G^*) < \Pi_{1,c}(G)$ and $\Pi_2(G^*) > \Pi_2(G)$, we are done. \square

Lemma 2 *Let G be a k -tree, if either $\Pi_{1,c}(G)$ attains the maximal or $\Pi_2(G)$ attains the minimal, then every hyper pendent edge is a k -path.*

Proof. Let $\mathcal{P} = G[V(S) \cup V(C)]$ be a hyper pendent edge, where $G[S] = G[\{x_1, x_2 \dots x_k\}]$ is a cut k -clique and $V(C) = \{u_1, u_2 \dots u_p\}$ with p is a positive integer such that u_1 is the only vertex of \mathcal{P} in $S_1(G)$ and for $i \in [1, p-1]$, u_i is the vertex added following by u_{i+1} through the process of Definition 1.

Fact 7. *For any hyper pendent edge $\mathcal{P} = G[V(S) \cup V(C)]$ as represented above, $\{u_1, u_2 \dots u_p\}$ is a simplicial elimination ordering of \mathcal{P} .*

Proof. By contradiction, assume that $\{u_1, u_2 \dots u_p\}$ is not a simplicial elimination ordering of \mathcal{P} . Let u_t be the first vertex from u_1 to u_p such that $\{u_t, u_{t+1}\} \in S_t$ for $t \in [2, p-1]$, then $u_t u_{t+1} \notin E(G)$ and $\{u_t, u_{t+1}\}$ can not be in some k -cliques. And by Definition 1, there must be at least two vertices that belongs to S_1 in $V(C)$, a contradiction. \square

By Fact 7, we know $\{u_1, u_2 \dots u_p\}$ is a simplicial elimination ordering of \mathcal{P} . For $p \leq 2$, \mathcal{P} is a k -path by Definition 2; For $p \geq 3$, if \mathcal{P} is a k -path, then we are done. Otherwise, let u_s be the first vertex from u_p to u_1 such that $G[V(S) \cup \{u_p, u_{p-1} \dots u_{s+1}, u_s\}]$ is not a k -path. Since $G[V(S) \cup \{u_p, u_{p-1} \dots u_{s+1}\}]$ is a k -path, for each $i \in [s+1, p]$, let $N_{G-\{u_1, u_2 \dots u_{i-1}\}}(u_i) = \{u_{i+1}, u_{i+2} \dots u_{\min\{p, i+k\}}, x_1, x_2 \dots x_{\max\{0, k-p+i\}}\}$, and by Definition 2 and the symmetry of $G[S]$, we have $|N(u_s) \cap \{u_{s+1}, u_{s+2} \dots u_{\min\{p, s+k\}}\}| = \min\{p-s-1, k-1\}$, where $1 \leq s \leq p-1$.

For $p \leq k+s$, suppose that u_t is the vertex such that $u_t \notin N(u_s)$ with $s+2 \leq t \leq p$, let $N_{G-\{u_1, u_2 \dots u_{s-1}\}}(u_s) = \{u_{s+1}, u_{s+2} \dots u_{t-1}, u_{t+1} \dots u_p, x_1, x_2 \dots x_{k-p+s+1}\}$, and let $|N(x_{k-p+s+1}) \cap \{u_1, u_2 \dots u_{s-1}\}| = m$ for $m \in [0, s-1]$. By Definition 2, we have $u_t u_i \notin E(G)$ for $i \in [1, s]$, and then $d(u_t) = k+t-s-1$ and $d(x_{k-p+s+1}) > k+p-s+m-1$. Now construct a new graph G^* such that $V(G^*) = V(G)$, $E(G^*) = E(G) - \{u_s x_{k-p+s+1}, u_i x_{k-p+s+1}\} + \{u_s u_t, u_i u_t\}$

with $i \in [0, m]$, then G^* is a k -tree. Since $t \leq p$, we have $d(x_{k-p+s+1}) > d(u_t) + m + 1$, and by the definition of $\Pi_{1,c}(G)$, $\Pi_2(G)$ and Fact 5, Fact 6, we get

$$\begin{aligned}
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{d(u_t)^c d(x_{k-p+s+1})^c}{[d(u_t) + m + 1]^c [d(x_{k-p+s+1}) - m - 1]^c} \\
&= \frac{\left[\frac{d(u_t)}{d(u_t) + m + 1}\right]^c}{\left[\frac{d(x_{k-p+s+1}) - m - 1}{d(x_{k-p+s+1})}\right]^c} \\
&< 1, \\
\frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{d(u_t)^{d(u_t)} d(x_{k-p+s+1})^{d(x_{k-p+s+1})}}{[d(u_t) + m + 1]^{d(u_t)+m+1} [d(x_{k-p+s+1}) - m - 1]^{d(x_{k-p+s+1})-m-1}} \\
&= \frac{\frac{[d(u_t) + m + 1]^{d(u_t)+m+1}}{[d((x_{k-p+s+1}) - m - 1)]^{d((x_{k-p+s+1})-m-1)}}}{d(x_{k-p+s+1})^{d(x_{k-p+s+1})}} \\
&> 1.
\end{aligned}$$

Thus, $\Pi_{1,c}(G^*) > \Pi_{1,c}(G)$ and $\Pi_2(G^*) < \Pi_2(G)$, a contradiction.

For $p \geq k + s + 1$, let $|N(u_{k+s+1}) \cap \{u_1, u_2 \dots u_{s-1}\}| = m$ for $m \in [0, s - 1]$. Since $G[V(S) \cup \{u_p, u_{p-1} \dots u_{s+1}\}]$ is a k -path, we have $G[\{u_{s+1}, u_{s+2} \dots u_{s+k+1}\}]$ is a $k + 1$ -clique. Suppose that u_t is the vertex such that $u_t \notin N(u_s)$ with $s + 2 \leq t \leq s + k$, let $N_{G-\{u_1, u_2 \dots u_{s-1}\}}(u_s) = \{u_{s+1}, u_{s+2} \dots u_{t-1}, u_{t+1} \dots u_{s+k+1}\}$. Now we construct a new graph G^* such that $V(G^*) = V(G)$, $E(G^*) = E(G) - \{u_s u_{k+s+1}, u_i u_{k+s+1}\} + \{u_s u_t, u_i u_t\}$ for $i \in [0, m]$, then G^* is a k -tree and $d(u_{k+s+1}) = 2k + m$, $d(u_t) = k + t - s - 1$. Since $t \leq s + k$, we have $d(u_{k+s+1}) > d(u_t) + m + 1$, and by the definition of $\Pi_{1,c}(G)$, $\Pi_2(G)$ and Fact 5, Fact 6, we get

$$\begin{aligned}
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{d(u_t)^c d(u_{k+s+1})^c}{[d(u_t) + m + 1]^c [d(u_{k+s+1}) - m - 1]^c} \\
&= \frac{\left[\frac{d(u_t)}{d(u_t) + m + 1}\right]^c}{\left[\frac{d(u_{k+s+1}) - m - 1}{d(u_{k+s+1})}\right]^c} \\
&< 1,
\end{aligned}$$

$$\begin{aligned}
\frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{d(u_t)^{d(u_t)} d(u_{k+s+1})^{d(u_{k+s+1})}}{[d(u_t) + m + 1]^{d(u_t)+m+1} [d(u_{k+s+1}) - m - 1]^{d(u_{k+s+1})-m-1}} \\
&= \frac{\frac{d(u_t)^{d(u_t)}}{[d(u_t) + m + 1]^{d(u_t)+m+1}}}{\frac{[d(u_{k+s+1}) - m - 1]^{d(u_{k+s+1})-m-1}}{d(u_{k+s+1})^{d(u_{k+s+1})}}} \\
&> 1.
\end{aligned}$$

Thus, $\Pi_{1,c}(G^*) > \Pi_{1,c}(G)$ and $\Pi_2(G^*) < \Pi_2(G)$, a contradiction. Hence, for any $s \in [1, p]$ $N_{G-\{u_1, u_2 \dots u_{s-1}\}}(u_s) = \{u_{s+1}, u_{s+2} \dots u_{\min\{p, k+s\}}, x_1, x_2 \dots x_{\max\{0, k-p+s\}}\}$, that is, \mathcal{P} is a k -tree. \square

Lemma 3 *Let G be a k -tree, if either $\Pi_{1,c}(G)$ attains the maximal or $\Pi_2(G)$ attains the minimal, then $|S_1(G)| = 2$.*

Proof. We know that $|S_1(G)| \geq 2$ for $n \geq k + 2$, and by Lemma 2, every hyper pendent edge is a k -path for $\Pi_{1,c}(G)$ to attain the maximal or $\Pi_2(G)$ to attain the minimal. If $|S_1(G)| = 2$, we are done; Suppose that $|S_1(G)| \geq 3$, it suffices to prove that there exists a graph G' such that $|S_1(G')| = |S_1(G)| - 1$ with $\Pi_{1,c}(G') > \Pi_{1,c}(G)$ and $\Pi_2(G') < \Pi_2(G)$.

Fact 8. *For any k -tree G satisfying the conditions of Lemma 3, if $|S_1(G)| \geq 3$, then there exists a k -clique $G[S]$ such that $w(G - S) \geq 3$.*

Proof. We will proceed by induction on $n = |G|$. For $n = k + 3$, it's trivial; For $n \geq k + 4$, assume that the fact is true for the k -tree G with $n < k + p$, and consider $n = k + p$. If $|S_1(G)| \geq 4$, choose any vertex $v \in S_1(G)$, or $|S_1(G)| = 3$ and $|S_2(G)| \geq 2$, choose the vertex $v \in S_1(G)$ such that $N(w) \cap S_1(G) = \{v\}$ for some $w \in S_2(G)$, then construct a new graph G' such that $G' = G - v$. Since $S_2(G)$ is an dependent set and $G[N(v)]$ is a k -clique for any $v \in S_1(G)$, we obtain $|S_1(G')| \geq 3$. By the induction hypothesis, there exists a k -clique $G[S]$ in G' such that $w(G' - S) \geq 3$. Thus, by adding back v , $G[S]$ is still a k -clique in G and $w(G - S) \geq 3$, we are done. Next, we only consider $|S_1(G)| = 3$ and $|S_2(G)| = 1$.

Let $S_1(G) = \{v_1, v_2, v_3\}$ and $G_0 = G - \{v_1, v_2, v_3\}$, by Definition 4, we have G_0 is a $k + 1$ -clique, denoted $G[\{x_1, x_2 \dots x_{k+1}\}]$. If there exists $N(v_i) = N(v_j)$, for some $i, j \in [1, 3]$ with $i \neq j$, and take $S = N(v_i)$, then $w(G - S) \geq 3$, we are done; If $N(v_i) \neq N(v_j)$, for any $i, j \in [1, 3]$ with $i \neq j$, then reorder the index of x_i such that $N(v_1) = \{x_1, x_2 \dots x_k\}$, $N(v_2) = \{x_2, x_3 \dots x_{k+1}\}$ and $N(v_3) = \{x_1, x_3 \dots x_{k+1}\}$. Construct a new graph G^* such that $V(G^*) = V(G)$, $E(G^*) = E(G) - \{v_1 x_1\} + \{v_1 v_2\}$, then G^* is still a k -tree and $d_G(x_1) = k + 2$, $d_{G^*}(x_1) = k + 1$, $d_G(v_1) = d_G(v_2) = k$ and $d_{G^*}(v_2) = k + 1$. By the definition of $\Pi_{1,c}(G)$, $\Pi_2(G)$ and Fact 6, we

have

$$\begin{aligned}
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{d(v_2)^c d(x_1)^c}{[d(v_2) + 1]^c [d(x_1) - 1]^c} \\
&= \left[\frac{k(k+2)}{(k+1)^2} \right]^c \\
&< 1, \\
\frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{d(v_2)^{d(v_2)} d(x_1)^{d(x_1)}}{[d(v_2) + 1]^{d(v_2)+1} [d(x_1) - 1]^{d(x_1)-1}} \\
&= \frac{(k+2)^{k+2} k^k}{(k+1)^{2(k+1)}} \\
&= \frac{k^k}{(k+1)^{k+1}} \\
&= \frac{[\frac{k^k}{(k+1)^{k+1}}]}{[\frac{(k+1)^{k+1}}{(k+2)^{k+2}}]} \\
&> 1.
\end{aligned}$$

Thus, we find a graph G^* with $\Pi_{1,c}(G^*) > \Pi_{1,c}(G)$ and $\Pi_2(G^*) < \Pi_2(G)$, a contradiction with that $\Pi_{1,c}(G)$ attains the maximal or $\Pi_2(G)$ attains the minimal, we are done. \square

Choose a k -clique $G[S]$ with $w(G - S) \geq 3$ such that there are two components of $G - S$: C_1, C_2 with $|C_1| = p, |C_2| = q$ and $p + q$ being minimal, for $p \geq q$. Let $u_1 \in V(C_1), v_1 \in V(C_2)$ with $\{u_1, v_1\} \subseteq S_1(G)$. Let $N_{G-\{v_1, v_2 \dots v_{i-1}\}}(v_i) = \{v_{i+1}, v_{i+2} \dots v_{\min\{k+1, q\}}, x_1, x_2 \dots x_{\max\{0, k-q+i\}}, N_{G-\{u_1, u_2 \dots u_{j-1}\}}(u_j) = \{u_{j+1}, u_{j+2} \dots u_{\min\{k+1, p\}}, y_1, y_2 \dots y_{\max\{0, k-p+i\}}\}$ for $i \geq 1, j \geq 1$, where $\{v_1, v_2 \dots v_q\}$ and $\{u_1, u_2 \dots u_p\}$ are simplicial elimination orderings of $G[S \cup V(C_1)]$ and $G[S \cup V(C_2)]$, respectively. We will prove Lemma 3 by induction on q .

(1) If $q = 1$, then $d(v_1) = k$. Choose $x_t \in N(v_1)$, let $|N(x_t) \cap \{u_1, u_2 \dots u_p\}| = m$ for $m \in [1, k]$, we get $d(x_t) > k + 1 + m$ by $w(G - S) \geq 3$, and then $d(x_t) > d(v_1) + m + 1$. Now construct a new graph G^* such that $V(G^*) = V(G), E(G^*) = E(G) - \{u_i x_t\} + \{u_i v_1\}$ for $i \in [1, m]$, then G^* is a k -tree and $|C_1| + |C_2| = p$ with $G[\{x_1, x_2 \dots x_{t-1}, x_{t+1} \dots x_k, v_1\}]$ is a k -clique in G^* . Since $d(x_t) > d(v_1) + m + 1$, by the definition of $\Pi_{1,c}(G), \Pi_2(G)$ and Fact 5, Fact 6, we have

$$\begin{aligned}
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{d(v_1)^c d(x_t)^c}{[d(v_1) + m]^c [d(x_t) - m]^c} \\
&= \frac{\left[\frac{d(v_1)}{d(v_1) + m} \right]^c}{\left[\frac{d(x_t) - m}{d(x_t)} \right]^c} \\
&< 1,
\end{aligned}$$

$$\begin{aligned}
\frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{d(v_1)^{d(v_1)} d(x_t)^{d(x_t)}}{[d(v_1) + m]^{d(v_1)+m} [d(x_t) - m]^{d(x_t)-m}} \\
&= \frac{\frac{d(v_1)^{d(v_1)}}{[d(v_1) + m]^{d(v_1)+m}}}{\frac{[d(x_t) - m]^{d(x_t)-m}}{d(x_t)^{d(x_t)}}} \\
&> 1.
\end{aligned}$$

Then, $\Pi_{1,c}(G^*) > \Pi_{1,c}(G)$ and $\Pi_2(G^*) < \Pi_2(G)$. Thus, let $G' = G^*$, $|S_1(G')| = |S_1(G)| - 1$, $\Pi_{1,c}(G') > \Pi_{1,c}(G)$ and $\Pi_2(G') < \Pi_2(G)$, and we are done.

(2) Assume that $q = s$, there exists a k -tree G' such that $|S_1(G')| = |S_1(G)| - 1$, $\Pi_{1,c}(G') > \Pi_{1,c}(G)$, $\Pi_2(G') < \Pi_2(G)$ and we consider $q = s + 1$.

If $q \leq k$, we have $d(v_q) = k + q - 1$ by the fact that $G[S \cup V(C_2)]$ is a k -path. Choose $x_t \in N(v_1)$, we know $x_t \in N(v_i)$ for all $i \in [1, p]$ by $G[S \cup V(C_2)]$ is a k -path. Let $|N(x_t) \cap \{u_1, u_2, \dots, u_p\}| = m$ for $m \in [1, k]$, we have $d(x_t) > k + q + m$ by $w(G - S) \geq 3$, and then $d(x_t) > d(v_q) + m + 1$. Now construct a new graph G^* such that $V(G^*) = V(G)$, $E(G^*) = E(G) - \{u_i x_t\} + \{u_i v_q\}$ for $i \in [1, m]$, then G^* is a k -tree and $|C_1| + |C_2| = p + q - 1$ with $G[\{x_1, x_2, \dots, x_{t-1}, x_{t+1}, \dots, x_k, v_q\}]$ is a k -clique in G^* . Since $d(x_t) > d(v_q) + m + 1$, by the definition of $\Pi_{1,c}(G)$, $\Pi_2(G)$ and Fact 5, Fact 6, we have

$$\begin{aligned}
\frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{d(v_q)^c d(x_t)^c}{[d(v_q) + m]^c [d(x_t) - m]^c} \\
&= \frac{\left[\frac{d(v_q)}{d(v_q) + m}\right]^c}{\left[\frac{d(x_t) - m}{d(x_t)}\right]^c} \\
&< 1,
\end{aligned}$$

$$\begin{aligned}
\frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{d(v_q)^{d(v_q)} d(x_t)^{d(x_t)}}{[d(v_q) + m]^{d(v_q)+m} [d(x_t) - m]^{d(x_t)-m}} \\
&= \frac{\frac{d(v_q)^{d(v_q)}}{[d(v_q) + m]^{d(v_q)+m}}}{\frac{[d(x_t) - m]^{d(x_t)-m}}{d(x_t)^{d(x_t)}}} \\
&> 1.
\end{aligned}$$

Then, $\Pi_{1,c}(G) < \Pi_{1,c}(G^*)$, $\Pi_2(G) > \Pi_2(G^*)$ and $q = s$ in G^* , then by the induction hypothesis, there exists a k -tree G' such that $|S_1(G')| = |S_1(G)| - 1$, $\Pi_{1,c}(G') > \Pi_{1,c}(G)$ and $\Pi_2(G') < \Pi_2(G)$, we are done.

If $q \geq k + 1$, we have $N(u_1) = \{u_2, u_3, \dots, u_{k+1}\}$, $N(v_1) = \{v_2, v_3, \dots, v_{k+1}\}$ by the facts that

$p \geq q$ and $G[S \cup V(C_1)]$, $G[S \cup V(C_2)]$ are k -paths. We construct a new graph G^* such that $V(G^*) = V(G)$, $E(G^*) = E(G) - \{v_1 v_i\} + \{u_j v_1\}$ for $i \in [2, k+1]$, $j \in [1, k]$. And by Fact 2 and the definition of $\Pi_{1,c}(G)$, $\Pi_2(G)$, we obtain

$$\begin{aligned} \frac{\Pi_{1,c}(G)}{\Pi_{1,c}(G^*)} &= \frac{\prod_{i=2}^{k+1} d(v_i)^c \prod_{j=1}^k d(u_j)^c}{\prod_{i=2}^{k+1} [d(v_i) - 1]^c \prod_{j=1}^k [d(u_j) + 1]^c} \\ &= 1, \\ \frac{\Pi_2(G)}{\Pi_2(G^*)} &= \frac{\prod_{i=2}^{k+1} d(v_i)^{d(v_i)} \prod_{j=1}^k d(u_j)^{d(u_j)}}{\prod_{i=2}^{k+1} [d(v_i) - 1]^{d(v_i)-1} \prod_{j=1}^k [d(u_j) + 1]^{d(u_j)+1}} \\ &= 1. \end{aligned}$$

Then, $\Pi_{1,c}(G) = \Pi_{1,c}(G^*)$, $\Pi_2(G) = \Pi_2(G^*)$ and $q = s$ in G^* , then by the induction hypothesis, there exists a k -tree G' such that $|S_1(G')| = |S_1(G)| - 1$, $\Pi_{1,c}(G') > \Pi_{1,c}(G)$ and $\Pi_2(G') < \Pi_2(G)$, we are done. \square

Now, we turn to prove the main results of the paper.

Proof of Theorem 2. For any k -tree T_n^k , if $|S_1(T_n^k)| = n - k$, then $T_n^k \cong S_{k,n-k}$, we are done. And if $|S_1(T_n^k)| \leq n - k - 1$, we can recursively use Lemma 1 to make $\Pi_{1,c}(T_n^k)$ decreasing until $|S_1(T_n^k)| = n - k$. Thus, we have $T_n^k \cong S_{k,n-k}$ for $\Pi_{1,c}(T_n^k)$ to arrive the minimal value.

By Lemma 2, if $\Pi_{1,c}(T_n^k)$ get the maximal, then every hyper pendent edge is a k -path, and by Lemma 3, $|S_1(T_n^k)| = 2$, implying that $T_n^k \cong P_n^k$ for $\Pi_{1,c}(T_n^k)$ to arrive the maximal value. \square

Proof of Theorem 3. For any k -tree T_n^k , if $|S_1(T_n^k)| = n - k$, then $T_n^k \cong S_{k,n-k}$, we are done. And if $|S_1(T_n^k)| \leq n - k - 1$, we can recursively use Lemma 1 to make $\Pi_2(T_n^k)$ increasing until $|S_1(T_n^k)| = n - k$, then we have $T_n^k \cong S_{k,n-k}$ for $\Pi_2(T_n^k)$ to arrive the maximal value.

By Lemma 2, if $\Pi_2(T_n^k)$ get the minimal, every hyper pendent edge is a k -path, and by Lemma 3, $|S_1(T_n^k)| = 2$. Then this k -tree is a k -path, that is, $T_n^k \cong P_n^k$ for $\Pi_2(T_n^k)$ to arrive the minimal value. \square

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